Chapter 1 Relations and Functions

EXERCISE 1.1

Question 1:

Determine whether each of the following relations are reflexive, symmetric and transitive.

(i) Relation R in the set $A=\{1,2,3,\ldots,13,14\}$ defined as

$$R=\{(x,y):3x-y=0\}$$

(ii) Relation R in the set of N natural numbers defined as

$$R=\{(x,y):y=x+5 \text{ and } x<4\}$$

(iii) Relation R in the set $A=\{1,2,3,4,5,6\}$ defined as

$$R=\{(x,y):y \text{ is divisible by } x\}$$

(iv) Relation R in the set of Z integers defined as

$$R=\{(x,y):x-y \text{ is an integer}\}$$

(v) Relation R in the set of human beings in a town at a particular time given by

(a) $R=\{(x,y):x \text{ and } y \text{ work at the same place}\}$

(b) $R=\{(x,y):x \text{ and } y \text{ live in the same locality}\}$

(c) $R=\{(x,y):x \text{ is exactly 7cm taller than } y\}$

(d) $R=\{(x,y):x \text{ is wife of } y\}$

(e) $R=\{(x,y):x \text{ is father of } y\}$

Solution:

(i) $R=\{(1,3),(2,6),(3,9),(4,12)\}$

R is not reflexive because $(1,1),(2,2)\ldots$ and $(14,14)\notin R$.

R is not symmetric because $(1,3)\in R$, but $(3,1)\notin R.[\text{since } 3(3)\neq 0]$.

R is not transitive because $(1,3),(3,9)\in R$, but $(1,9)\notin R.[3(1)-9\neq 0]$.

Hence, R is neither reflexive nor symmetric nor transitive.

(ii) $R=\{(1,6),(2,7),(3,8)\}$

R is not reflexive because $(1,1)\notin R$.

R is not symmetric because $(1,6)\in R$ but $(6,1)\notin R$.

R is not transitive because there isn't any ordered pair in R such that $(x,y),(y,z)\in R$, so $(x,z)\notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

(iii) $R=\{(x,y):y \text{ is divisible by } x\}$

We know that any number other than 0 is divisible by itself.

Thus, $(x,x)\in R$

So, R is reflexive.

$(2,4) \in R$ [because 4 is divisible by 2]
But $(4,2) \notin R$ [since 2 is not divisible by 4]
So, R is not symmetric.
Let (x,y) and $(y,z) \in R$. So, y is divisible by x and z is divisible by y.
So, z is divisible by x $\Rightarrow (x,z) \in R$
So, R is transitive.
So, R is reflexive and transitive but not symmetric.

(iv) $R = \{(x,y) : x - y \text{ is an integer}\}$
For $x \in Z, (x,x) \notin R$ because $x - x = 0$ is an integer.
So, R is reflexive.
For, $x,y \in Z$, if $x,y \in R$, then $x - y$ is an integer $\Rightarrow (y-x)$ is an integer.
So, $(y,x) \in R$
So, R is symmetric.
Let (x,y) and $(y,z) \in R$, where $x,y,z \in Z$.
$\Rightarrow (x-y)$ and $(y-z)$ are integers.
$\Rightarrow x - z = (x-y)+(y-z)$ is an integer.
So, R is transitive.
So, R is reflexive, symmetric and transitive.

(v)

a) $R = \{(x,y) : x \text{ and } y \text{ work at the same place}\}$
R is reflexive because $(x,x) \in R$
R is symmetric because ,
If $(x,y) \in R$, then x and y work at the same place and y and x also work at the same place. $(y,x) \in R$.
R is transitive because,
Let $(x,y),(y,z) \in R$
x and y work at the same place and y and z work at the same place.
Then, x and z also works at the same place. $(x,z) \in R$.
Hence, R is reflexive, symmetric and transitive.

b) $R = \{(x,y) : x \text{ and } y \text{ live in the same locality}\}$
R is reflexive because $(x,x) \in R$
R is symmetric because,
If $(x,y) \in R$, then x and y live in the same locality and y and x also live in the same locality $(y,x) \in R$.
R is transitive because,

Let $(x,y),(y,z)\in R$

x and y live in the same locality and y and z live in the same locality.

Then x and z also live in the same locality. $(x,z)\in R$.

Hence, R is reflexive, symmetric and transitive.

c) $R=\{(x,y): x \text{ is exactly 7cm taller than } y\}$

R is not reflexive because $(x,x)\notin R$.

R is not symmetric because,

If $(x,y)\in R$, then x is exactly $7cm$ taller than y and y is clearly not taller than x. $(y,x)\notin R$.

R is not transitive because,

Let $(x,y),(y,z)\in R$

x is exactly $7cm$ taller than y and y is exactly $7cm$ taller than z.

Then x is exactly $14cm$ taller than z. $(x,z)\notin R$

Hence, R is neither reflexive nor symmetric nor transitive.

d) $R=\{(x,y): x \text{ is wife of } y\}$

R is not reflexive because $(x,x)\notin R$.

R is not symmetric because,

Let $(x,y)\in R$, x is the wife of y and y is not the wife of x. $(y,x)\notin R$.

R is not transitive because,

Let $(x,y),(y,z)\in R$

x is wife of y and y is wife of z, which is not possible.

$(x,z)\notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

e) $R=\{(x,y): x \text{ is father of } y\}$

R is not reflexive because $(x,x)\notin R$.

R is not symmetric because,

Let $(x,y)\in R$, x is the father of y and y is not the father of x. $(y,x)\notin R$.

R is not transitive because,

Let $(x,y),(y,z)\in R$

x is father of y and y is father of z, x is not father of z. $(x,z)\notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

Question 2:

Show that the relation R in the set R of real numbers, defined as $R=\{(a,b): a \le b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution:

$R=\{(a,b): a \le b^2\}$

$\left(\frac{1}{2},\frac{1}{2}\right) \notin R$ because $\frac{1}{2} > \left(\frac{1}{2}\right)^2$

R is not reflexive.

$(1,4) \in R$ as $1<4$. But 4 is not less than 1^2.

$(4,1) \notin R$

R is not symmetric.

$(3,2)(2,1.5) \in R$ [Because $3<2^2=4$ and $2<(1.5)^2=2.25$]

$3>(1.5)^2=2.25$

$\therefore (3,1.5) \notin R$

R is not transitive.

R is neither reflective nor symmetric nor transitive.

Question 3:

Check whether the relation R defined in the set $\{1,2,3,4,5,6\}$ as $R=\{(a,b): b=a+1\}$ is reflexive, symmetric or transitive.

Solution:

$A=\{1,2,3,4,5,6\}$

$R=\{(a,b): b=a+1\}$

$R=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

$(a,a) \notin R, a \in A$

$(1,1),(2,2),(3,3),(4,4),(5,5) \notin R$

R is not reflexive.

$(1,2) \in R$, but $(2,1) \notin R$

R is not symmetric.

$(1,2),(2,3) \in R$

$(1,3) \notin R$

R is not transitive.

R is neither reflective nor symmetric nor transitive.

Question 4:

Show that the relation R in R defined as $R = \{(a,b) : a \le b\}$ is reflexive and transitive, but not symmetric.

Solution:

$R = \{(a,b) : a \le b\}$

$(a,a) \in R$

R is reflexive.

$(2,4) \in R$ (as $2 < 4$)

$(4,2) \notin R$ (as $4>2$)

R is not symmetric.

$(a,b),(b,c) \in R$

$a \le b$ and $b \le c$

$\Rightarrow a \le c$

$\Rightarrow (a,c) \in R$

R is transitive.

R is reflexive and transitive but not symmetric.

Question 5:

Check whether the relation R in R defined as $R = \{(a,b) : a \le b^3\}$ is reflexive, symmetric or transitive.

Solution:

$R = \{(a,b) : a \le b^3\}$

$\left(\frac{1}{2},\frac{1}{2}\right) \notin R$, since $\frac{1}{2} > \left(\frac{1}{2}\right)^3$

R is not reflexive.

$(1,2) \in R\left(\text{as } 1 < 2^3 = 8\right)$

$(2,1) \notin R\left(\text{as } 2^3 > 1 = 8\right)$

R is not symmetric.

$\left(3,\frac{3}{2}\right),\left(\frac{3}{2},\frac{6}{5}\right) \in R,$ since $3 < \left(\frac{3}{2}\right)^3$ and $\frac{2}{3} < \left(\frac{6}{2}\right)^3$

$\left(3,\frac{6}{5}\right) \notin R\, 3 > \left(\frac{6}{5}\right)^3$

R is not transitive.

R is neither reflexive nor symmetric nor transitive.

Question 6:

Show that the relation R in the set $\{1,2,3\}$ given by $R = \{(1,2),(2,1)\}$ is symmetric but neither reflexive nor transitive.

Solution:

$A = \{1,2,3\}$

$R = \{(1,2),(2,1)\}$

$(1,1),(2,2),(3,3) \notin R$

R is not reflexive.

$(1,2) \in R$ and $(2,1) \in R$

R is symmetric.

$(1,2) \in R$ and $(2,1) \in R$

$(1,1) \in R$

R is not transitive.

R is symmetric, but not reflexive or transitive.

Question 7:

Show that the relation R in the set A of all books in a library of a college, given by $R = \{(x,y): x \text{ and } y \text{ have same number of pages}\}$ is an equivalence relation.

Solution:

$R = \{(x,y): x \text{ and } y \text{ have same number of pages}\}$

R is reflexive since $(x,x) \in R$ as x and x have same number of pages.

R is reflexive.

$(x, y) \in R$

x and y have same number of pages and y and x have same number of pages $(y, x) \in R$
R is symmetric.

$(x, y) \in R, (y, z) \in R$
x and y have same number of pages, y and z have same number of pages.
Then x and z have same number of pages.
$(x, z) \in R$
R is transitive.

R is an equivalence relation.

Question 8:

Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation. Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

Solution:
$a \in A$
$|a - a| = 0 (\text{which is even})$

R is reflective.

$(a, b) \in R$
$\Rightarrow |a - b| \text{ [is even]}$
$\Rightarrow |-(a - b)| = |b - a| \text{ [is even]}$
$(b, a) \in R$
R is symmetric.

$(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow |a - b|$ is even and $|b - c|$ is even
$\Rightarrow (a - b)$ is even and $(b - c)$ is even
$\Rightarrow (a - c) = (a + b) + (b - c)$ is even

$\Rightarrow |a-b|$ is even

$\Rightarrow (a,c) \in R$

R is transitive.

R is an equivalence relation.

All elements of $\{1,3,5\}$ are related to each other because they are all odd. So, the modulus of the difference between any two elements is even.

Similarly, all elements $\{2,4\}$ are related to each other because they are all even.

No element of $\{1,3,5\}$ is related to any elements of $\{2,4\}$ as all elements of $\{1,3,5\}$ are odd and all elements of $\{2,4\}$ are even. So, the modulus of the difference between the two elements will not be even.

Question 9:

Show that each of the relation R in the set $A=\{x \in Z : 0 \le x \le 12\}$, given by

i. $R=\{(a,b):|a-b| \text{ is a mutiple of } 4\}$

ii. $R=\{(a,b): a=b\}$

Is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution:

$A=\{x \in Z : 0 \le x \le 12\}=\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$

i. $R=\{(a,b):|a-b| \text{ is a mutiple of } 4\}$

$a \in A, (a,a) \in R \quad [|a-a|=0 \text{ is a multiple of } 4]$

R is reflexive.

$(a,b) \in R \Rightarrow |a-b| \; [\text{is a multiple of } 4]$

$\Rightarrow |-(a-b)|=|b-a| \; [\text{is a multiple of } 4]$

$(b,a) \in R$

R is symmetric.

$(a,b) \in R$ and $(b,c) \in$ R

$\Rightarrow |a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4

$\Rightarrow (a-b)$ is a multiple of 4 and $(b-c)$ is a multiple of 4

$\Rightarrow (a-c)=(a-b)+(b-c)$ is a multiple of 4

$\Rightarrow |a-c|$ is a multiple of 4

$\Rightarrow (a,c) \in R$

R is transitive.

R is an equivalence relation.

The set of elements related to 1 is $\{1,5,9\}$ as

$|1-1| = 0$ is a multiple of 4.

$|5-1| = 4$ is a multiple of 4.

$|9-1| = 8$ is a multiple of 4.

ii. $R = \{(a,b) : a = b\}$

$a \in A, (a,a) \in R$ [since a=a]

R is reflective.

$(a,b) \in R$

$\Rightarrow a = b$

$\Rightarrow b = a$

$\Rightarrow (b,a) \in R$

R is symmetric.

$(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow a = b$ and $b = c$

$\Rightarrow a = c$

$\Rightarrow (a,c) \in R$

R is transitive.

R is an equivalence relation.

The set of elements related to 1 is $\{1\}$.

Question 10:

Give an example of a relation, which is

i. Symmetric but neither reflexive nor transitive.
ii. Transitive but neither reflexive nor symmetric.
iii. Reflexive and symmetric but not transitive.
iv. Reflexive and transitive but not symmetric.
v. Symmetric and transitive but not reflexive.

Solution:

i.

$A=\{5,6,7\}$

$R=\{(5,6),(6,5)\}$

$(5,5),(6,6),(7,7)\notin R$

R is not reflexive as $(5,5),(6,6),(7,7)\notin R$

$(5,6),(6,5)\in R$ and $(6,5)\in R$, R is symmetric.

$\Rightarrow(5,6),(6,5)\in R$, but $(5,5)\notin R$

R is not transitive.

Relation R is symmetric but not reflexive or transitive.

ii. $R=\{(a,b):a<b\}$

$a\in R,(a,a)\notin R$ [since a cannot be less than itself]

R is not reflexive.

$(1,2)\in R(\text{as } 1<2)$

But 2 is not less than 1

$\therefore(2,1)\notin R$

R is not symmetric.

$(a,b),(b,c)\in R$

$\Rightarrow a<b$ and $b<c$

$\Rightarrow a<c$

$\Rightarrow(a,c)\in R$

R is transitive.

Relation R is transitive but not reflexive and symmetric.

iii. $A=\{4,6,8\}$

$A=\{(4,4),(6,6),(8,8),(4,6),(6,8),(8,6)\}$

R is reflexive since $a\in A,(a,a)\in R$

R is symmetric since $(a,b)\in R$

$\Rightarrow(b,a)\in R \quad for\ a,b\in R$

R is not transitive since $(4,6),(6,8)\in R, but\ (4,8)\notin R$

R is reflexive and symmetric but not transitive.

iv. $R=\{(a,b):a^3>b^3\}$

$(a,a)\in R$

R is reflexive.

$(2,1)\in R$

$But\ (1,2)\notin R$

$\therefore$ R is not symmetric.

$(a,b),(b,c) \in R$
$\Rightarrow a^3 \geq b^3$ and $b^3 < c^3$
$\Rightarrow a^3 < c^3$
$\Rightarrow (a,c) \in R$
$\therefore$ R is transitive.
R is reflexive and transitive but not symmetric

v.

$A = \{1, 3, 5\}$ Define a Relation R

On A.

$R : A \rightarrow A$

$R = \{(1,3)\ (3,1)\ (1,1)\ (3,3)\}$

Relation R is not Reflexive as $(5, 5) \not\subset R$

Relation R is Symmetric as

$(1,3) \in R \Rightarrow (3,1) \in R$

Relation R is Transitive

$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$

$(3,1) \in R, (1,1) \in R \Rightarrow (3,1) \in R$

Alternative Answer

$R = (a, b) : a$ is brother of b {suppose a and b are male}

$Ref \rightarrow a$ is not brother of a

So, $(a, a) \not\subset R$

Relation R is not Reflexive

Symmetric$\rightarrow a$ is brother of b so

b is brother of a

$a, b \in \mathrm{R} \Rightarrow (b, a) \in \mathrm{R}$

Transitive$\rightarrow a$ is brother of b and

b is brother of c so

a is brother of c

$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$

Question 11:

Show that the relation R in the set A of points in a plane given by

$R=\{(P,Q):\text{Distance of the point P from the origin is same as the distance of the point Q from the origin}\}$

, is an equivalence relation. Further, show that the set of all points related to a point $P\neq(0,0)$ is the circle passing through P with origin as centre.

Solution:

$R=\{(P,Q):\text{Distance of the point P from the origin is same as the distance of the point Q from the origin}\}$

Clearly, $(P,P)\in R$

R is reflexive.

$(P,Q)\in R$

Clearly R is symmetric.

$(P,Q),(Q,S)\in R$

$\Rightarrow$ The distance of P and Q from the origin is the same and also, the distance of Q and S from the origin is the same.

$\Rightarrow$ The distance of P and S from the origin is the same.

$(P,S)\in R$

R is transitive.

R is an equivalence relation.

The set of points related to $P\neq(0,0)$ will be those points whose distance from origin is same as distance of P from the origin.

Set of points forms a circle with the centre as origin and this circle passes through P.

Question 12:

Show that the relation R in the set A of all triangles as $R=\{(T_1,T_2):T_1 \text{ is similar to } T_2\}$, is an equivalence relation. Consider three right angle triangles T_1 with sides $3,4,5$, T_2 with sides $5,12,13$ and T_3 with sides $6,8,10$. Which triangle among T_1,T_2,T_3 are related?

Solution:

$R=\{(T_1,T_2):T_1 \text{ is similar to } T_2\}$

R is reflexive since every triangle is similar to itself.

If $(T_1,T_2)\in R$, then T_1 is similar to T_2.

T_2 is similar to T_1.

$\Rightarrow(T_2,T_1)\in R$

R is symmetric.

$(T_1,T_2),(T_2,T_3)\in R$

is similar to T_2 and T_2 is similar to T_3.

$\therefore$ T_1 is similar to T_3.

$\Rightarrow (T_1, T_3) \in R$

$\therefore$ R is transitive.

$$\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \left(\frac{1}{2}\right)$$

$\therefore$ Corresponding sides of triangles T_1 and T_3 are in the same ratio.

Triangle T_1 is similar to triangle T_3.

Hence, T_1 is related to T_3.

Question 13:

Show that the relation R in the set A of all polygons as $R = \{(P_1, P_2) : P_1 \text{and } P_2 \text{ have same number of sides}\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides $3, 4 and 5$?

Solution:

$R = \{(P_1, P_2) : P_1 \text{and } P_2 \text{ have same number of sides}\}$

$(P_1, P_2) \in R$ as same polygon has same number of sides.

$\therefore$ R is reflexive.

$(P_1, P_2) \in R$

$\Rightarrow P_1$ and P_2 have same number of sides.

$\Rightarrow P_2$ and P_1 have same number of sides.

$\Rightarrow (P_2, P_1) \in R$

$\therefore$ R is symmetric.

$(P_1, P_2), (P_2, P_3) \in R$

$\Rightarrow P_1$ and P_2 have same number of sides.

P_2 and P_3 have same number of sides.

$\Rightarrow P_1$ and P_3 have same number of sides.

$\Rightarrow (P_1, P_3) \in R$

$\therefore$ R is transitive.

R is an equivalence relation.

The elements in A related to right-angled triangle (T) with sides $3, 4, 5$ are those polygons which have three sides.

Set of all elements in a related to triangle T is the set of all triangles.

Question 14:

Let L be the set of all lines in XY plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1 \text{is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Solution:

$R = \{(L_1, L_2) : L_1 \text{is parallel to } L_2\}$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_2) \in R$

If $(L_1, L_2) \in R$, then

$\Rightarrow L_1$ is parallel to L_2.

$\Rightarrow L_2$ is parallel to L_1.

$\Rightarrow (L_2, L_1) \in R$
$\therefore$ R is symmetric.

$(L_1, L_2), (L_2, L_3) \in R$
$\Rightarrow L_1$ is parallel to L_2
$\Rightarrow L_2$ is parallel to L_3
$\therefore L_1$ is parallel to L_3.
$\Rightarrow (L_1, L_3) \in R$
$\therefore$ R is transitive.

R is an equivalence relation.

Set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.
Slope of the line $y = 2x + 4$ is $m = 2$.
Line parallel to the given line is in the form $y = 2x + c$, where $c \in R$.
Set of all lines related to the given line is given by $y = 2x + c$, where $c \in R$.

Question 15:

Let R be the relation in the set $\{1, 2, 3.4\}$ given by
$R = \{(1,2)(2,2),(1,1),(4,4),(1,3),(3,3),(3,2)\}$.
Choose the correct answer.

A. R is reflexive and symmetric but not transitive.
B. R is reflexive and transitive but not symmetric.
C. R is symmetric and transitive but not reflexive.
D. R is an equivalence relation.

Solution:

$R = \{(1,2)(2,2),(1,1),(4,4),(1,3),(3,3),(3,2)\}$
$(a,a) \in R$ for every $a \in \{1,2,3.4\}$
$\therefore$ R is reflexive.

$(1,2) \in R$ but $(2,1) \notin R$
$\therefore$ R is not symmetric.

$(a,b),(b,c) \in R$ for all $a,b,c \in \{1,2,3,4\}$
$\therefore$ R is not transitive.

R is reflexive and transitive but not symmetric.

The correct answer is B.

Question 16:

Let R be the relation in the set N given by $R=\{(a,b):a=b-2,b>6\}$. Choose the correct answer.

A. $(2,4)\in R$

B. $(3,8)\in R$

C. $(6,8)\in R$

D. $(8,7)\in R$

Solution:

$R=\{(a,b):a=b-2,b>6\}$

Now,

$b>6,(2,4)\notin R$

$3\neq 8-2$

$\therefore (3,8)\notin R$ and as $8\neq 7-2$

$\therefore (8,7)\notin R$

Consider $(6,8)$

$8>6$ and $6=8-2$

$\therefore (6,8)\in R$

The correct answer is C.

EXERCISE 1.2

Question 1:

Show that the function $f: R_* \to R_*$ defined by $(x)=\frac{1}{x}$ is one –one and onto, where R_* is the set of all non –zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Solution:

$f: R_* \to R_*$ is by $f(x)=\frac{1}{x}$

For one-one:

$x, y \in R_*$ such that $f(x)=f(y)$

$\Rightarrow \frac{1}{x}=\frac{1}{y}$

$\Rightarrow x=y$

$\therefore f$ is one-one.

For onto:

For $y \in R$, there exists $x=\frac{1}{y} \in R_* \left[\text{as } y \notin 0\right]$ such that

$f(x)=\frac{1}{\left(\frac{1}{y}\right)}=y$

$\therefore f$ is onto.

Given function f is one-one and onto.

Consider function $g: N \to R$, defined by $g(x)=\frac{1}{x}$

We have, $g(x_1)=g(x_2) \Rightarrow \frac{1}{x_1}=\frac{1}{x_2} \Rightarrow x_1 = x_2$

$\therefore g$ is one-one.

g is not onto as for $1.2 \in R$, there exist any x in N such that $g(x)=\frac{1}{1.2}$

Function g is one-one but not onto.

Question 2:

Check the injectivity and surjectivity of the following functions:

i. $f: N \to N$ given by $f(x)=x^2$

ii. $f: Z \to Z$ given by $f(x)=x^2$

iii. $f: R \to R$ given by $f(x)=x^2$

iv. $f: N \to N$ given by $f(x)=x^3$

v. $f: Z \to Z$ given by $f(x)=x^3$

Solution:

i. For $f: N \to N$ given by $f(x)=x^2$

$x, y \in N$

$f(x)=f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$

$\therefore f$ is injective.

$2 \in N$. But, there does not exist any x in N such that $f(x) = x^2 = 2$
$\therefore f$ is not surjective
Function f is injective but not surjective.

ii. $f : Z \to Z$ given by $f(x) = x^2$
$f(-1) = f(1) = 1$ but $-1 \neq 1$
$\therefore f$ is not injective.

$-2 \in Z$ But, there does not exist any $x \in Z$ such that $f(x) = -2 \Rightarrow x^2 = -2$
$\therefore f$ is not surjective.
Function f is neither injective nor surjective.

iii. $f : R \to R$ given by $f(x) = x^2$
$f(-1) = f(1) = 1$ but $-1 \neq 1$
$\therefore f$ is not injective.

$-2 \in Z$ But, there does not exist any $x \in Z$ such that $f(x) = -2 \Rightarrow x^2 = -2$
$\therefore f$ is not surjective.
Function f is neither injective nor surjective.

iv. $f : N \to N$ given by $f(x) = x^3$
$x, y \in N$
$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$
$\therefore f$ is injective.

$2 \in N$. But, there does not exist any x in N such that $f(x) = x^3 = 2$
$\therefore f$ is not surjective
Function f is injective but not surjective.

v. $f : Z \to Z$ given by $f(x) = x^3$
$x, y \in Z$
$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$
$\therefore f$ is injective.

$2 \in Z$. But, there does not exist any x in Z such that $f(x) = x^3 = 2$
$\therefore f$ is not surjective.
Function f is injective but not surjective.

Question 3:

Prove that the greatest integer function $f: R \to R$ given by $f(x)=[x]$ is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Solution:

$f: R \to R$ given by $f(x)=[x]$

$f(1.2)=[1.2]=1, f(1.9)=[1.9]=1$

$\therefore f(1.2)=f(1.9)$, but $1.2 \neq 1.9$

$\therefore f$ is not one-one.

Consider $0.7 \in R$

$f(x)=[x]$ is an integer. There does not exist any element $x \in R$ such that $f(x)=0.7$

$\therefore f$ is not onto.

The greatest integer function is neither one-one nor onto.

Question 4:

Show that the modulus function $f: R \to R$ given by $f(x)=|x|$ is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Solution:

$f: R \to R$ is $f(x)=|x|=\begin{cases} \text{x, if x} \geq 0 \\ -x, \text{if x} < 0 \end{cases}$

$f(-1)=|-1|=1$ and $f(1)=|1|=1$

$\therefore f(-1)=f(1)$ but $-1 \neq 1$

$\therefore f$ is not one-one.

Consider $-1 \in R$

$f(x)=|x|$ is non-negative. There exist any element x in domain R such that $f(x)=|x|=-1$

$\therefore f$ is not onto.

The modulus function is neither one-one nor onto.

Question 5:

Show that the signum function $f: R \rightarrow R$ given by $f(x)=\begin{cases} 1, \text{ if } x>0 \\ 0, \text{ if } x=0 \\ -1, \text{ if } x<0 \end{cases}$ is neither one-one nor onto.

Solution:

$f: R \rightarrow R$ is $f(x)=\begin{cases} 1, \text{if } x>0 \\ 0, \text{ if } x=0 \\ -1, \text{ if } x<0 \end{cases}$

$f(1)=f(2)=1$, but $1 \neq 2$

$\therefore f$ is not one-one.

$f(x)$ takes only 3 values $(1,0,-1)$ for the element -2 in co-domain R, there does not exist any x in domain R such that $f(x)=-2$.

$\therefore f$ is not onto.

The signum function is neither one-one nor onto.

Question 6:

Let $A=\{1,2,3\}$, $B=\{4,5,6,7\}$ and let $f=\{(1,4),(2,5),(3,6)\}$ be a function from A to B. Show that f is one-one.

Solution:

$A=\{1,2,3\}$, $B=\{4,5,6,7\}$

$f: A \rightarrow B$ is defined as $f=\{(1,4),(2,5),(3,6)\}$

$\therefore f(1)=4, f(2)=5, f(3)=6$

It is seen that the images of distinct elements of A under f are distinct.

$\therefore f$ is one-one.

Question 7:

In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

i. $f: R \rightarrow R$ defined by $f(x)=3-4x$

ii. $f: R \rightarrow R$ defined by $f(x)=1+x^2$

Solution:

i. $f : R \to R$ defined by $f(x) = 3 - 4x$

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 3 - 4x_1 = 3 - 4x_2$

$\Rightarrow -4x = -4x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

For any real number (y) in R, there exists $\frac{3-y}{4}$ in R such that $f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y$

$\therefore f$ is onto.

Hence, f is bijective.

ii. $f : R \to R$ defined by $f(x) = 1 + x^2$

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 1 + x_1^2 = 1 + x_2^2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

Consider $f(1) = f(-1) = 2$

$\therefore f$ is not one-one.

Consider an element -2 in co domain R.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in R$.

$\therefore f$ is not onto.

Hence, f is neither one-one nor onto.

Question 8:

Let A and B be sets. Show that $f : A \times B \to B \times A$ such that $(a,b) = (b,a)$ is a bijective function.

Solution:

$f : A \times B \to B \times A$ is defined as $(a,b) = (b,a)$.

$(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$
$\Rightarrow b_1 = b_2$ and $a_1 = a_2$
$\Rightarrow (a_1, b_1) = (a_2, b_2)$
$\therefore f$ is one-one.
$(b,a) \in B \times A$ there exist $(a,b) \in A \times B$ such that $f(a,b) = (b,a)$
$\therefore f$ is onto.
f is bijective.

Question 9:

Let $f : N \rightarrow N$ be defined as $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{if } n \text{ is even} \end{cases}$ for all $n \in N$. State whether the function f is bijective. Justify your answer.

Solution:

$f : N \rightarrow N$ be defined as $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{if } n \text{ is even} \end{cases}$ for all $n \in N$.

$f(1) = \frac{1+1}{2} = 1$ and $f(2) = \frac{2}{2} = 1$

$f(1) = f(2)$, where $1 \neq 2$

$\therefore f$ is not one-one.

Consider a natural number n in co domain N.

Case I: n is odd

$\therefore n = 2r + 1$ for some $r \in N$ there exists $4r + 1 \in N$ such that

$f(4r+1) = \frac{4r+1+1}{2} = 2r + 1$

Case II: n is even

$\therefore n = 2r$ for some $r \in N$ there exists $4r \in N$ such that

$f(4r) = \frac{4r}{2} = 2r$

$\therefore f$ is onto.

f is not a bijective function.

Question 10:

Let $A = R - \{3\}, B = R - \{1\}$ and $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

Solution:

$A = R - \{3\}, B = R - \{1\}$ and $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$

$x, y \in A$ such that $f(x) = f(y)$

$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$

$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$

$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$

$\Rightarrow -3x - 2y = -3y - 2x$

$\Rightarrow 3x - 2x = 3y - 2y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

Let $y \in B = R - \{1\}$, then $y \neq 1$

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now,

$f(x) = y$

$\Rightarrow \frac{x-2}{x-3} = y$

$\Rightarrow x - 2 = xy - 3y$

$\Rightarrow x(1-y) = -3y + 2$

$\Rightarrow x = \frac{2-3y}{1-y} \in A \qquad [y \neq 1]$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right) - 2}{\left(\frac{2-3y}{1--y}\right) - 3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

$\therefore f$ is onto.

Hence, the function is one-one and onto.

Question 11:

Let $f: R \to R$ defined as $f(x) = x^4$ .Choose the correct answer.

A. f is one-one onto

B. f is many-one onto

C. f is one-one but not onto

D. f is neither one-one nor onto

Solution:

$f: R \to R$ defined as $f(x) = x^4$

$x, y \in R$ such that $f(x) = f(y)$

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

$\therefore f(x) = f(y)$ does not imply that $x = y$.

For example $f(1) = f(-1) = 1$

$\therefore f$ is not one-one.

Consider an element 2 in co domain R there does not exist any x in domain R such that $f(x) = 2$.

$\therefore f$ is not onto.

Function f is neither one-one nor onto.
The correct answer is D.

Question 12:

Let $f: R \to R$ defined as $f(x) = 3x$ .Choose the correct answer.

A. f is one-one onto

B. f is many-one onto

C. f is one-one but not onto

D. f is neither one-one nor onto

Solution:

$f: R \to R$ defined as $f(x) = 3x$

$x, y \in R$ such that $f(x) = f(y)$

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

For any real number y in co domain R, there exist $\frac{y}{3}$ in R such that $f\left(\frac{y}{3}\right)=3\left(\frac{y}{3}\right)=y$

$\therefore f$ is onto.

Hence, function f is one-one and onto.

The correct answer is A.

EXERCISE 1.3

Question 1:

Let $f:\{1,3,4\}\to\{1,2,5\}$ and $g:\{1,2,5\}\to\{1,3\}$ be given by $f=\{(1,2),(3,5),(4,1)\}$ and $g=\{(1,3),(2,3),(5,1)\}$. Write down gof.

Solution:

The functions $f:\{1,3,4\}\to\{1,2,5\}$ and $g:\{1,2,5\}\to\{1,3\}$ are $f=\{(1,2),(3,5),(4,1)\}$ and $g=\{(1,3),(2,3),(5,1)\}$

$gof(1)=g[f(1)]=g(2)=3 \qquad [as\ f(1)=2\ and\ g(2)=3]$

$gof(3)=g[f(3)]=g(5)=1 \qquad [as\ f(3)=5\ and\ g(5)=1]$

$gof(4)=g[f(4)]=g(1)=3 \qquad [as\ f(4)=1\ and\ g(1)=3]$

$\therefore gof=\{(1,3),(3,1),(4,3)\}$

Question 2:

Let f,g,h be functions from R to R. Show that

$(f+g)oh=foh+goh$

$(f.g)oh=(foh).(goh)$

Solution:

$(f+g)oh=foh+goh$

$LHS=[(f+g)oh](x)$

$=(f+g)[h(x)]=f[h(x)]+g[h(x)]$

$=(foh)(x)+goh(x)$

$=\{(foh)+(goh)\}(x)=RHS$

$\therefore \{(f+g)oh\}(x)=\{(foh)+(goh)\}(x) \quad for\ all\ x\in R$

Hence, $(f+g)oh=foh+goh$

$(f.g)oh=(foh).(goh)$

$LHS=[(f.g)oh](x)$

$=(f.g)[h(x)]=f[h(x)].g[h(x)]$

$=(foh)(x).(goh)(x)$

$=\{(foh).(goh)\}(x)=RHS$

$\therefore [(f.g)oh](x)=\{(foh).(goh)\}(x) \quad for\ all\ x\in R$

Hence, $(f.g)oh=(foh).(goh)$

Question 3:

Find gof and fog, if

i. $f(x)=|x|$ and $g(x)=|5x-2|$

ii. $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$

Solution:

i. $f(x)=|x|$ and $g(x)=|5x-2|$

$\therefore gof(x)=g(f(x))=g(|x|)=|5|x|-2|$

$fog(x)=f(g(x))=f(|5x-2|)=||5x-2||=|5x-2|$

ii. $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$

$\therefore gof(x)=g(f(x))=g(8x^3)=(8x^3)^{\frac{1}{3}}=2x$

$fog(x)=f(g(x))=f\left(x^{\frac{1}{3}}\right)^3=8\left(x^{\frac{1}{3}}\right)^3=8x$

Question 4:

If $f(x)=\dfrac{(4x+3)}{(6x-4)}, x\neq\dfrac{2}{3}$, show that $fof(x)=x,\ for\ all\ x\neq\dfrac{2}{3}$. What is the reverse of f?

Solution:

$$(fof)(x)=f(f(x))=f\left(\frac{4x+3}{6x-4}\right)$$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{16x+12+18x-12}{24x+18-24x+16}=\frac{34x}{34}=x$$

$\therefore fof(x)=x \quad for\ all\ x\neq\dfrac{2}{3}$

$\Rightarrow fof=1$

Hence, the given function f is invertible and the inverse of f is f itself.

Question 5:

State with reason whether the following functions have inverse.

i. $f:\{1,2,3,4\}\to\{10\}$ with $f=\{(1,10),(2,10),(3,10),(4,10)\}$

ii. $g:\{5,6,7,8\}\to\{1,2,3,4\}$ with $g=\{(5,4),(6,3),(7,4),(8,2)\}$

iii. $h:\{2,3,4,5\}\to\{7,9,11,13\}$ with $h=\{(2,7),(3,9),(4,11),(5,13)\}$

Solution:

i. $f:\{1,2,3,4\}\to\{10\}$ with $f=\{(1,10),(2,10),(3,10),(4,10)\}$

f is a many one function as $f(1)=f(2)=f(3)=f(4)=10$

$\therefore f$ is not one-one.

Function f does not have an inverse.

ii. $g:\{5,6,7,8\}\to\{1,2,3,4\}$ with $g=\{(5,4),(6,3),(7,4),(8,2)\}$

g is a many one function as $g(5)=g(7)=4$

$\therefore g$ is not one-one.

Function g does not have an inverse.

iii. $h:\{2,3,4,5\}\to\{7,9,11,13\}$ with $h=\{(2,7),(3,9),(4,11),(5,13)\}$

All distinct elements of the set $\{2,3,4,5\}$ have distinct images under h.

$\therefore h$ is one-one.

h is onto since for every element y of the set $\{7,9,11,13\}$, there exists an element x in the set $\{2,3,4,5\}$, such that $h(x)=y$.

h is a one-one and onto function.

Function h has an inverse.

Question 6:

Show that $f:[-1,1]\to R$, given by $f(x)=\dfrac{x}{(x+2)}$ is one-one. Find the inverse of the function $f:[-1,1]\to Range\,f$.

(Hint: For $y\in Range\,f, y=f(x)=\dfrac{x}{x+2}$, for some x in $[-1,1]$, i,e., $x=\dfrac{2y}{(1-y)}$

Solution:

$f:[-1,1]\to R$, given by $f(x)=\dfrac{x}{(x+2)}$

For one-one

$f(x)=f(y)$

$\Rightarrow \dfrac{x}{x+2}=\dfrac{y}{y+2}$

$\Rightarrow xy+2x=xy+2y$

$\Rightarrow 2x=2y$

$\Rightarrow x=y$

$\therefore f$ is a one-one function.

It is clear that $f:[-1,1]\to R$ is onto.

$f:[-1,1]\to R$ is one-one and onto and therefore, the inverse of the function $f:[-1,1]\to R$ exists.

Let $g: Range\, f \to [-1,1]$ be the inverse of f.

Let y be an arbitrary element of range f.

Since $f:[-1,1]\to Range\, f$ is onto, we have:

$y=f(x)\;\; for\;\; same\;\; x\in[-1,1]$

$\Rightarrow y=\dfrac{x}{x+2}$

$\Rightarrow xy+2y=x$

$\Rightarrow x(1-y)=2y$

$\Rightarrow x=\dfrac{2y}{1-y}, y\neq 1$

Now, let us define $g: Range\, f \to [-1,1]$ as

$g(y)=\dfrac{2y}{1-y}, y\neq 1$

Now,

$$(gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(fog)(x) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

$\therefore gof = I_{[-1,1]} \quad and \quad fog = I_{Range\, f}$

$\therefore f^{-1} = g$

$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$

Question 7:

Consider $f : R \to R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

Solution:

$f : R \to R$ given by $f(x) = 4x + 3$

For one-one

$f(x) = f(y)$

$\Rightarrow 4x + 3 = 4y + 3$

$\Rightarrow 4x = 4y$

$\Rightarrow x = y$

$\therefore f$ is a one-one function.

For onto

$y \in R,\ let\ y = 4x + 3$

$\Rightarrow x = \frac{y-3}{4} \in R$

Therefore, for any $y \in R$, there exists $x = \frac{y-3}{4} \in R$ such that

$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y$

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g : R \to R$ by $g(x) = \frac{y-3}{4}$

Now,

$$(gof)(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = x$$

$$(fog)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

$$\therefore gof = fog = I_R$$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \frac{y-3}{4}$.

Question 8:

Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with inverse f^{-1} of given f by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Solution:

$f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$

For one-one:

Let $f(x) = f(y)$

$\Rightarrow x^2 + 4 = y^2 + 4$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y \qquad [as\ x \in R]$

$\therefore f$ is a one -one function.

For onto:

For $y \in [4, \infty),\ let\ y = x^2 + 4$

$\Rightarrow x^2 = y - 4 \geq 0 \qquad [as\ y \geq 4]$

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in R$, there exists $x = \sqrt{y-4} \in R$ such that

$$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y$$

$\therefore f$ is an onto function.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g : [4, \infty) \to R_+$ by

$g(y)=\sqrt{y-4}$

Now, $gof(x)=g(f(x))=g(x^2+4)=\sqrt{(x^2+4)-4}=\sqrt{x^2}=x$

And $fog(y)=f(g(y))=f(\sqrt{y-4})=(\sqrt{y-4})^2+4=(y-4)+4=y$

$\therefore gof=fog=I_R$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y)=g(y)=\sqrt{y-4}$.

Question 9:

Consider $f:R_+\to[-5,\infty)$ given by $f(x)=9x^2+6x-5$. Show that f is invertible with $f^{-1}(y)=\left(\frac{(\sqrt{y+6})-1}{3}\right)$.

Solution:

$f:R_+\to[-5,\infty)$ given by $f(x)=9x^2+6x-5$

Let y be an arbitrary element of $[-5,\infty)$.

Let $y=9x^2+6x-5$

$\Rightarrow y=(3x+1)^2-1-5$

$\Rightarrow y=(3x+1)^2-6$

$\Rightarrow (3x+1)^2=y+6$

$\Rightarrow 3x+1=\sqrt{y+6}$ $\quad [as\ y\geq -5\Rightarrow y+6>0]$

$\Rightarrow x=\frac{\sqrt{y+6}-1}{3}$

$\therefore f$ is onto, thereby range $f=[-5,\infty)$.

Let us define $g:[-5,\infty)\to R_+\ as\ g(y)=\frac{\sqrt{y+6}-1}{3}$

We have,

$$(gof)(x) = g(f(x)) = g(9x^2 + 6x - 5)$$
$$= g\left((3x+1)^2 - 6\right)$$
$$= \frac{\sqrt{(3x+1)^2 - 6 + 6} - 1}{3}$$
$$= \frac{3x + 1 - 1}{3} = x$$

And,

$$(fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6} - 1}{3}\right)$$
$$= \left[3\left(\frac{\sqrt{y+6} - 1}{3}\right) + 1\right]^2 - 6$$
$$= \left(\sqrt{y+6}\right)^2 - 6 = y + 6 - 6 = y$$
$$\therefore gof = I_R \quad and \quad fog = I_{[-5,\infty)}$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6} - 1}{3}.$$

Question 10:

Let $f : X \to Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$, $fog_1(y) = I_Y(y) = fog_2(y)$. Use one-one ness of f.

Solution:

Let $f : X \to Y$ be an invertible function.

Also suppose f has two inverses (g_1 and g_2)

Then, for all $y \in Y$,

$fog_1(y) = I_Y(y) = fog_2(y)$

$\Rightarrow f(g_1(y)) = f(g_2(y))$

$\Rightarrow g_1(y) = g_2(y)$ $\quad$ [f is invertible $\Rightarrow f$ is one-one]

$\Rightarrow g_1 = g_2$ $\quad$ [g is one-one]

Hence, f has unique inverse.

Question 11:

Consider $f:\{1,2,3\}\rightarrow\{a,b,c\}$ given by $f(1)=a, f(2)=b, f(3)=c$. Find $\left(f^{-1}\right)^{-1}=f$.

Solution:

Function $f:\{1,2,3\}\rightarrow\{a,b,c\}$ given by $f(1)=a, f(2)=b, f(3)=c$

If we define $g:\{a,b,c\}\rightarrow\{1,2,3\}$ as $g(a)=1, g(b)=2, g(c)=3$

$(fog)(a)=f(g(a))=f(1)=a$

$(fog)(b)=f(g(b))=f(2)=b$

$(fog)(c)=f(g(c))=f(3)=c$

And,

$(gof)(1)=g(f(1))=g(a)=1$

$(gof)(2)=g(f(2))=g(b)=2$

$(gof)(3)=g(f(3))=g(c)=3$

$\therefore gof=I_X$ and $fog=I_Y$ where $X=\{(1,2,3)\}$ and $Y=\{a,b,c\}$

Thus, the inverse of f exists and $f^{-1}=g$.

$\therefore f^{-1}:\{a,b,c\}\rightarrow\{1,2,3\}$ is given by, $f^{-1}(a)=1, f^{-1}(b)=2, f^{-1}(c)=3$

We need to find the inverse of f^{-1} i.e., inverse of g.

If we define $h:\{1,2,3\}\rightarrow\{a,b,c\}$ as $h(1)=a, h(2)=b, h(3)=c$

$(goh)(1)=g(h(1))=g(a)=1$

$(goh)(2)=g(h(2))=g(b)=2$

$(goh)(3)=g(h(3))=g(c)=3$

And,

$(hog)(a)=h(g(a))=h(1)=a$

$(hog)(b)=h(g(b))=h(2)=b$

$(hog)(c)=h(g(c))=h(3)=c$

$\therefore goh=I_X$ and $hog=I_Y$ where $X=\{(1,2,3)\}$ and $Y=\{a,b,c\}$

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow \left(f^{-1}\right)^{-1} = h$.
It can be noted that $h = f$.

Hence, $\left(f^{-1}\right)^{-1} = f$

Question 12:

Let $f : X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f i.e., $\left(f^{-1}\right)^{-1} = f$.

Solution:

Let $f : X \to Y$ be an invertible function.

Then there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$

Here, $f^{-1} = g$

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, $f^{-1} : Y \to X$ is invertible and f^{-1} is f i.e., $\left(f^{-1}\right)^{-1} = f$.

Question 13:

If $f : R \to R$ is given by $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$, then $fof(x)$ is:

A. $\frac{1}{x^3}$
B. x^3
C. x
D. $\left(3 - x^3\right)$

Solution:

$f : R \to R$ is given by $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$

$f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$

$$\therefore fof(x) = f(f(x)) = f\left(\left(3 - x^3\right)^{\frac{1}{3}}\right) = \left[3 - \left(\left(3 - x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[3 - \left(3 - x^3\right)\right]^{\frac{1}{3}} = \left(x^3\right)^{\frac{1}{3}} = x$$

$\therefore fof(x) = x$

The correct answer is C.

Question 14:

If $f:R-\left\{-\frac{4}{3}\right\}\to R$ be a function defined as $f(x)=\frac{4x}{3x+4}$. The inverse of f is the map $g:Range\,f\to R-\left\{-\frac{4}{3}\right\}$ given by :

A. $g(y)=\frac{3y}{3-4y}$

B. $g(y)=\frac{4y}{4-3y}$

C. $g(y)=\frac{4y}{3-4y}$

D. $g(y)=\frac{3y}{4-3y}$

Solution:

It is given that $f:R-\left\{-\frac{4}{3}\right\}\to R$ is defined as $f(x)=\frac{4x}{3x+4}$

Let y be an arbitrary element of Range f.

Then, there exists $x\in R-\left\{-\frac{4}{3}\right\}$ such that $y=f(x)$.

$\Rightarrow y=\frac{4x}{3x+4}$

$\Rightarrow 3xy+4y=4x$

$\Rightarrow x(4-3y)=4y$

$\Rightarrow x=\frac{4y}{4-3y}$

Define $f:R-\left\{-\frac{4}{3}\right\}\to R$ as $g(y)=\frac{4y}{4-3y}$

Now,

$$(gof)(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$$

$$= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x}$$

$$= \frac{16x}{16} = x$$

And

$$(fog)(x) = (g(x)) = f\left(\frac{4y}{4-3y}\right)$$

$$= \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y}$$

$$= \frac{16y}{16} = y$$

$\therefore gof = I_{R-\left\{-\frac{4}{3}\right\}}$ and $fog = I_{Range\ f}$

Thus, g is the inverse of f i.e., $f^{-1} = g$

Hence, the inverse of f is the map $g : Range\ f \to R - \left\{-\frac{4}{3}\right\}$, which is given by $g(y) = \frac{4y}{4-3y}$.

The correct answer is B.

EXERCISE 1.4

Question 1:

Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.

i. On $\mathbf{Z}^+$, define $*$ by $a*b=a-b$
ii. On $\mathbf{Z}^+$,define $*$ by $a*b=ab$
iii. On $\mathbf{R}$, define $*$by $a*b=ab^2$
iv. On $\mathbf{Z}^+$, define $*$ by $a*b=|a-b|$
v. On $\mathbf{Z}^+$, define $*$ by $a*b=a$

Solution:

i. On $\mathbf{Z}^+$, define $*$ by $a*b=a-b$

It is not a binary operation as the image of $(1,2)$ under $*$ is

$1*2=1-2$

$\Rightarrow -1 \notin \mathbf{Z}^+$.

Therefore, $*$ is not a binary operation.

ii. On $\mathbf{Z}^+$,define $*$ by $a*b=ab$

It is seen that for each $a,b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.

This means that $*$ carries each pair (a,b) to a unique element $a*b=ab$ in $\mathbf{Z}^+$.

Therefore, $*$ is a binary operation.

iii. On $\mathbf{R}$, define $*$ $a*b=ab^2$

It is seen that for each $a,b \in \mathbf{R}$, there is a unique element ab^2 in $\mathbf{R}$. This means that $*$ carries each pair (a,b) to a unique element $a*b=ab^2$ in $\mathbf{R}$.

Therefore, $*$is a binary operation.

iv. On $\mathbf{Z}^+$, define $*$ by $a*b=|a-b|$

It is seen that for each $a,b \in \mathbf{Z}^+$,there is a unique element $|a-b|$ in $\mathbf{Z}^+$. This means that $*$ carries each pair (a,b) to a unique element $a*b=|a-b|$ in $\mathbf{Z}^+$. Therefore, $*$is a binary operation.

v. On $\mathbf{Z}^+$, define $*$ by $a*b=a$

$*$carries each pair (*a*, *b*) to a unique element in $a*b=a$ in $\mathbf{Z}^+$.

Therefore, $*$ is a binary operation.

Question 2:

For each binary operation $*$defined below, determine whether $*$ is commutative or associative.

i. On $\mathbf{Z}^+$, define $a*b=a-b$

ii. On $\mathbf{Q}$, define $a*b=ab+1$

iii. On $\mathbf{Q}$, define $a*b=\frac{ab}{2}$

iv. On $\mathbf{Z}^+$, define $a*b=2^{ab}$

v. On $\mathbf{Z}^+$, define $a*b=a^b$

vi. On $\mathbf{R}-\{-1\}$, define $a*b=\frac{a}{b+1}$

Solution:

i. On $\mathbf{Z}^+$, define $a*b=a-b$

It can be observed that $1*2=1-2=-1$ and $2*1=2-1=1$.

$\therefore 1*2\neq 2*1$; where $1,\ 2\in\mathbf{Z}$

Hence, the operation $*$ is not commutative.

Also,

$(1*2)*3=(1-2)*3=-1*3=-1-3=-4$

$1*(2*3)=1*(2-3)=1*-1=1-(-1)=2$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in\mathbf{Z}$

Hence, the operation $*$ is not associative.

ii. On $\mathbf{Q}$, define $a*b=ab+1$

$ab=ba$ for all $a,b\in Q$

$\Rightarrow ab+1=ba+1$ for all $a,b\in Q$

$\Rightarrow a*b=b*a$ for all $a,b\in Q$

Hence, the operation $*$ is commutative.

$(1*2)*3=(1\times 2+1)*3=3*3=3\times 3+1=10$

$1*(2*3)=1*(2\times 3+1)=1*7=1\times 7+1=8$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in\mathbf{Q}$

Hence, the operation $*$ is not associative.

iii. On $\mathbf{Q}$, define $a*b=\frac{ab}{2}$

$ab=ba$ for all $a,b\in Q$

$\Rightarrow \frac{ab}{2}=\frac{ab}{2}$ for all $a,b\in Q$

$\Rightarrow a*b=b*a$ for all $a,b\in Q$

Hence, the operation $*$ is commutative.

$$(a*b)*c=\left(\frac{ab}{2}\right)*c=\frac{\left(\frac{ab}{2}\right)c}{2}=\frac{abc}{4}$$

And

$$a*(b*c)=a*\left(\frac{bc}{2}\right)=\frac{a\left(\frac{bc}{2}\right)}{2}=\frac{abc}{4}$$

$\therefore (a*b)*c=a*(b*c)$ where $a,b,c\in\mathbf{Q}$

Hence, the operation $*$ is associative.

iv. On $\mathbf{Z}^+$, define $a*b=2^{ab}$

$ab=ba$ for all $a,b\in Z$

$\Rightarrow 2^{ab}=2^{ba}$ for all $a,b\in Z$

$\Rightarrow a*b=b*a$ for all $a,b\in Z$

Hence, the operation $*$ is commutative.

$(1*2)*3=2^{1\times2}*3=4*3=2^{4\times3}=2^{12}$

$1*(2*3)=1*2^{2\times3}=1*2^6=1*64=2^{64}$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in\mathbf{Z}^+$

Hence, the operation $*$ is not associative.

v. On $\mathbf{Z}^+$, define $a*b=a^b$

$1*2=1^2=1$

$2*1=2^1=2$

$\therefore 1*2\neq 2*1$ where $1,2,\in\mathbf{Z}^+$

Hence, the operation $*$ is not commutative.

$(2*3)*4=2^3*4=8*4=8^4=2^{12}$

$2*(3*4)=2*3^4=2*81=2^{81}$

$\therefore (2*3)*4\neq 2*(3*4)$ where $2,3,4\in\mathbf{Z}^+$

Hence, the operation $*$ is not associative.

vi. On $\mathbf{R}-\{-1\}$, define $a*b=\frac{a}{b+1}$

$$1*2=\frac{1}{2+1}=\frac{1}{3}$$

$$2*1=\frac{2}{1+1}=\frac{2}{2}=1$$

$\therefore 1*2 \neq 2*1$ where $1,2, \in \mathbf{R}-\{-1\}$

Hence, the operation $*$ is not commutative.

$$(1*2)*3 = \frac{1}{3}*3 = \frac{\frac{1}{3}}{3+1} = \frac{1}{12}$$

$$1*(2*3) = 1*\frac{2}{3+1} = 1*\frac{2}{4} = 1*\frac{1}{2} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$\therefore (1*2)*3 \neq 1*(2*3)$ where $1,2,3 \in \mathbf{R}-\{-1\}$

Hence, the operation $*$ is not associative.

Question 3:

Consider the binary operation $\wedge$ on the set $\{1,2,3,4,5\}$ defined by $a \wedge b = \min\{a,b\}$. Write the operation table of the operation $\wedge$.

Solution:

The binary operation $\wedge$ on the set $\{1,2,3,4,5\}$ is defined by $a \wedge b = \min\{a,b\}$ for all $a,b \in \{1,2,3,4,5\}$.

The operation table for the given operation $\wedge$ can be given as:

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Question 4:

Consider a binary operation $*$ on the set $\{1,2,3,4,5\}$ given by the following multiplication table.

i. Compute $(2*3)*4$ and $2*(3*4)$

ii. Is $*$ commutative?

iii. Compute $(2*3)*(4*5)$.

(Hint: Use the following table)

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1

3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Solution:

$(2*3)*4 = 1*4 = 1$

i. $2*(3*4) = 2*1 = 1$

ii. For every $a, b \in \{1,2,3,4,5\}$, we have $a*b = b*a$. Therefore, $*$ is commutative.

iii. $(2*3)*(4*5)$

$(2*3) = 1$ *and* $(4*5) = 1$

$\therefore (2*3)*(4*5) = 1*1 = 1$

Question 5:

Let $*'$ be the binary operation on the set $\{1,2,3,4,5\}$ defined by $a*'b =$ H.C.F. of a and b. Is the operation $*'$ same as the operation $*$ defined in Exercise 4 above? Justify your answer.

Solution:

The binary operation on the set $\{1,2,3,4,5\}$ is defined by $a*'b =$ H.C.F. of a and b.
The operation table for the operation $*'$ can be given as:

$*'$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

The operation table for the operations $*'$ and $*$ are same.
operation $*'$ is same as operation $*$.

Question 6:

Let $*$ be the binary operation on N defined by $a*b =$ L.C.M. of a and b. Find

i. $5*7, 20*16$

ii. Is $*$ commutative?

iii. Is $*$ associative?

iv. Find the identity of $*$ in N

v. Which elements of N are invertible for the operation $*$?

Solution:

The binary operation on N is defined by $a*b=$ L.C.M. of a and b.

i. $5*7=$L.C.M of 5 and $7=35$

$20*16=$LCM of 20 and $16=80$

ii. L.C.M. of a and b=LCM of b and a for all $a,b\in N$

$\therefore a*b=b*a$

Operation $*$ is commutative.

iii. For $a,b,c\in N$

$(a*b)*c=$ (L.C.M. of a and b)$*c=$ L.C.M. of a,b,c

$a*(b*c)=a*$(L.C.M. of b and c)= L.C.M. of a,b,c

$\therefore (a*b)*c=a*(b*c)$

Operation $*$ is associative.

iv. L.C.M. of a and $1=a=$ L.C.M. of 1 and a for all $a\in N$

$a*1=a=1*a$ for all $a\in N$

Therefore, 1 is the identity of $*$ in N.

v. An element a in N is invertible with respect to the operation $*$ if there exists an element b in N, such that $a*b=e=b*a$

$e=1$

L.C.M. of a and $b=1$=LCM of b and a possible only when a and b are equal to 1.

1 is the only invertible element of N with respect to the operation $*$.

Question 7:

Is $*$ defined on the set $\{1,2,3,4,5\}$ by $a*b=$ LCM of a and b a binary operation? Justify your answer.

Solution:

The operation $*$ on the set $\{1,2,3,4,5\}$ is defined by $a*b=$ LCM of a and b.

The operation table for the operation $*'$ can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

$3*2=2*3=6\notin A,$
$5*2=2*5=10\notin A,$
$3*4=4*3=12\notin A,$
$3*5=5*3=15\notin A,$
$4*5=5*4=20\notin A$
The given operation $*$ is not a binary operation.

Question 8:

Let $*$ be the binary operation on N defined by $a*b$= H.C.F. of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N?

Solution:

The binary operation $*$ on N defined by $a*b$= H.C.F. of a and b.
$\therefore a*b=b*a$
Operation $*$ is commutative.

For all $a,b,c\in N$,
$(a*b)*c$ = (HCF of a and b) $*c$ = HCF of a,b,c
$a*(b*c)$ = $a*$(HCF. of b and c)= HCF of a,b,c
$\therefore (a*b)*c=a*(b*c)$
Operation $*$ is associative.

$e\in N$ will be the identity for the operation $*$ if $a*e=a=e*a$ for all $a\in N$. But this relation is not true for any $a\in N$.
Operation $*$ does not have any identity in N.

Question 9:

Let $*$ be the binary operation on Q of rational numbers as follows:

i. $a*b=a-b$
ii. $a*b=a^2+b^2$
iii. $a*b=a+ab$
iv. $a*b=(a-b)^2$
v. $a+b=\frac{ab}{4}$
vi. $a*b=ab^2$

Find which of the binary operations are commutative and which are associative.

Solution:

i. On Q, the operation * is defined as $a*b=a-b$

$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}-\frac{1}{3}=\frac{3-2}{3}=\frac{1}{6}$

And

$\frac{1}{3}*\frac{1}{2}=\frac{1}{3}-\frac{1}{2}=\frac{2-3}{6}=\frac{-1}{6}$

$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)\neq\left(\frac{1}{3}*\frac{1}{2}\right)$ where $\frac{1}{2},\frac{1}{3}\in Q$

Operation * is not commutative.

$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left(\frac{1}{2}-\frac{1}{3}\right)*\frac{1}{4}=\frac{1}{6}*\frac{1}{4}=\frac{1}{6}-\frac{1}{4}=\frac{2-3}{12}=\frac{-1}{12}$

$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{1}{2}*\frac{1}{12}=\frac{1}{2}-\frac{1}{12}=\frac{6-1}{12}=\frac{5}{12}$

$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)$ where $\frac{1}{2},\frac{1}{3},\frac{1}{4}\in Q$

Operation * is not associative.

ii. On Q, the operation * is defined as $a*b=a^2+b^2$

For $a,b\in Q$

$a*b=a^2+b^2=b^2+a^2=b*a$

$\therefore a*b=b*a$

Operation * is commutative.

$(1*2)*3=(1^2+2^2)*3=(1+4)*3=5*3=5^2+3^2=25+9=34$

$1*(2*3)=1*(2^2+3^2)=1*(4+9)=1*13=1^2+13^2=1+169=170$

$\therefore(1*2)*3\neq1*(2*3)$ where $1,2,3\in Q$

Operation * is not associative.

iii. On Q, the operation * is defined as $a*b=a+ab$

$1*2=1+1\times2=1+2=3$

$2*1=2+2\times1=2+2=4$

$\therefore 1*2\neq2*1$ where $1,2\in Q$

Operation * is not commutative.

$(1*2)*3=(1+1\times2)*3=3*3=3+3\times3=3+9=12$

$1*(2*3)=1*(2+2\times3)=1*8=1+1\times8=1+8=9$

$\therefore(1*2)*3\neq1*(2*3)$ where $1,2,3\in Q$

Operation * is not associative.

iv. On Q, the operation * is defined as $a*b=(a-b)^2$

For $a,b\in Q$

$a*b=(a-b)^2$

$b*a=(b-a)^2=[-(a-b)]^2=(a-b)^2$

$\therefore a*b=b*a$

Operation * is commutative.

$(1*2)*3=(1-2)^2*3=(-1)^2*3=1*3=(1-3)^2=(-2)^2=4$

$1*(2*3)=1*(2-3)^2=1*(-1)^2=1*1=(1-1)^2=0$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in Q$

Operation * is not associative.

v. On Q, the operation * is defined as $a+b=\frac{ab}{4}$

For $a,b\in Q$

$a*b=\frac{ab}{4}=\frac{ba}{4}=b*a$

$\therefore a*b=b*a$

Operation * is commutative.

For $a,b,c\in Q$

$(a*b)*c=\frac{ab}{4}*c=\frac{\frac{ab}{4}\cdot c}{4}=\frac{abc}{16}$

$a*(b*c)=a*\frac{ab}{4}=\frac{a\cdot\frac{ab}{4}}{4}=\frac{abc}{16}$

$\therefore (a*b)*c=a*(b*c)$ where $a,b,c\in Q$

Operation * is associative.

vi. On Q, the operation * is defined as $a*b=ab^2$

$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2=\frac{1}{2}\cdot\frac{1}{9}=\frac{1}{18}$

$\frac{1}{3}*\frac{1}{2}=\frac{1}{3}\cdot\left(\frac{1}{2}\right)^2=\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}$

$\therefore \left(\frac{1}{2}*\frac{1}{3}\right)\neq\left(\frac{1}{3}*\frac{1}{2}\right)$ where $\frac{1}{2},\frac{1}{3}\in Q$

Operation * is not commutative.

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left(\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2\right)*\frac{1}{4}=\frac{1}{18}*\frac{1}{4}=\frac{1}{18}\cdot\left(\frac{1}{4}\right)^2=\frac{1}{18\times 16}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left(\frac{1}{3}\cdot\left(\frac{1}{4}\right)^2\right)=\frac{1}{2}*\frac{1}{48}=\frac{1}{2}\cdot\left(\frac{1}{48}\right)^2=\frac{1}{2\times(48)^2}$$

$$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right) \quad \text{where } \frac{1}{2},\frac{1}{3},\frac{1}{4}\in Q$$

Operation $*$ is not associative.

Operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

Question 10:

Find which of the operations given above has identity.

Solution:

An element $e \in Q$ will be the identity element for the operation $*$ if

$a*e = a = e*a,$ for all $a \in Q$

$$a*b=\frac{ab}{4}$$

$$\Rightarrow a*e=a$$

$$\Rightarrow \frac{ae}{4}=a$$

$$\Rightarrow e=4$$

Similarly, it can be checked for $e*a=a$, we get $e=4$ is the identity.

Question 11:

$A = N \times N$ and $*$ be the binary operation on A defined by $(a,b)*(c,d)=(a+c,b+d)$. Show that $*$ is commutative and associative. Find the identity element for $*$ on A, if any.

Solution:

$A = N \times N$ and $*$ be the binary operation on A defined by

$(a,b)*(c,d)=(a+c,b+d)$

$(a,b)*(c,d)\in A$

$a,b,c,d\in N$

$(a,b)*(c,d)=(a+c,b+d)$

$(c,d)*(a,b)=(c+a,d+b)=(a+c,b+d)$

$\therefore (a,b)*(c,d)=(c,d)*(a,b)$

Operation $*$ is commutative.

Now, let $(a,b),(c,d),(e,f)\in A$

$a,b,c,d,e,f\in N$

$[(a,b)*(c,d)]*(e,f)=(a+c,b+d)*(e,f)=(a+c+e,b+d+f)$

$(a,b)*[(c,d)*(e,f)]=(a,b)*(c+e,d+f)=(a+c+e,b+d+f)$

$\therefore [(a,b)*(c,d)]*(e,f)=(a,b)*[(c,d)*(e,f)]$

Operation * is associative.

An element $e=(e_1,e_2)\in A$ will be an identity element for the operation * if $a+e=a=e*a$ for all $a=(a_1,a_2)\in A$ i.e., $(a_1+e_1,a_2+e_2)=(a_1,a_2)=(e_1+a_1,e_2+a_2)$, which is not true for any element in A.

Therefore, the operation * does not have any identity element.

Question 12:

State whether the following statements are true or false. Justify.

i. For an arbitrary binary operation * on a set N, $a*a=a$ for all $a\in N$.

ii. If * is a commutative binary operation on N, then $a*(b*c)=(c*b)*a$

Solution:

i. Define operation * on a set N as $a*a=a$ for all $a\in N$.

In particular, for $a=3$,

$3*3=9\neq 3$

Therefore, statement (i) is false.

ii. R.H.S. = $(c*b)*a$

$=(b*c)*a$ [* is commutative]

$=a*(b*c)$ [Again, as * is commutative]

= L.H.S.

$\therefore a*(b*c)=(c*b)*a$

Therefore, statement (ii) is true.

Question 13:

Consider a binary operation * on N defined as $a*b=a^3+b^3$. Choose the correct answer.

A. Is * both associative and commutative?

B. Is * commutative but not associative?

C. Is * associative but not commutative?

D. Is * neither commutative nor associative?

Solution:

On N, operation $*$ is defined as $a*b=a^3+b^3$.

For all $a,b\in N$

$a*b=a^3+b^3=b^3+a^3=b*a$

Operation $*$ is commutative.

$(1*2)*3=(1^3+2^3)*3=(1+8)*3=9*3=9^3+3^3=729+27=756$

$1*(2*3)=1*(2^3+3^3)=1*(8+27)=1*35=1^3+35^3=1+42875=42876$

$\therefore (1*2)*3\neq 1*(2*3)$ Operation $*$ is not associative.

Therefore, Operation $*$ is commutative, but not associative.

The correct answer is B.

MISCELLANEOUS EXERCISE

Question 1:

Let $f: R \rightarrow R$ be defined as $f(x)=10x+7$. Find the function $g: R \rightarrow R$ such that $gof = fog = I_R$.

Solution:

$f: R \rightarrow R$ is defined as $f(x)=10x+7$

For one-one:

$f(x)=f(y)$ where $x, y \in R$

$\Rightarrow 10x+7=10y+7$

$\Rightarrow x=y$

$\therefore f$ is one-one.

For onto:

$y \in R$, Let $y=10x+7$

$\Rightarrow x=\frac{y-7}{10} \in R$

For any $y \in R$, there exists $x=\frac{y-7}{10} \in R$ such that

$f(x)=f\left(\frac{y-7}{10}\right)=10\left(\frac{y-7}{10}\right)+7=y-7+7=y$

$\therefore f$ is onto.

Thus, f is an invertible function.

Let us define $g: R \rightarrow R$ as $g(y)=\frac{y-7}{10}$.

Now,

$$gof(x)=g(f(x))=g(10x+7)=\frac{(10x+7)-7}{10}=\frac{10x}{10}=10$$

And,

$$fog(y)=f(g(y))=f\left(\frac{y-7}{10}\right)=10\left(\frac{y-7}{10}\right)+7=y-7+7=y$$

$\therefore gof = I_R$ and $fog = I_R$

Hence, the required function $g: R \rightarrow R$ as $g(y)=\frac{y-7}{10}$.

Question 2:

Let $f: W \to W$ be defined as $f(n) = n-1$, if is odd and $f(n) = n+1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Solution:

$f: W \to W$ is defined as $f(n) = \begin{cases} n-1, \text{ If } n \text{ is odd} \\ n+1, \text{ If } n \text{ is even} \end{cases}$

For one-one:

$f(n) = f(m)$

If n is odd and m is even, then we will have $n-1 = m+1$.

$\Rightarrow n - m = 2$

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

∴Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$f(n) = f(m)$

$\Rightarrow n - 1 = m - 1$

$\Rightarrow n = m$

Again, if both n and m are even, then we have:

$f(n) = f(m)$

$\Rightarrow n + 1 = m + 1$

$\Rightarrow n = m$

$\therefore f$ is one-one.

For onto:

Any odd number $2r+1$ in co-domain N is the image of $2r$ in domain N and any even number $2r$ in co-domain N is the image of $2r+1$ in domain N.

$\therefore f$ is onto.

f is an invertible function.

Let us define $g: W \to W$ as $f(m) = \begin{cases} m-1, \text{ If } m \text{ is odd} \\ m+1, \text{ If } m \text{ is even} \end{cases}$

When r is odd

$gof(n) = g(f(n)) = g(n-1) = n-1+1 = n$

When r is even

$gof(n) = g(f(n)) = g(n+1) = n+1-1 = n$

When m is odd

$fog(n) = f(g(m)) = f(m-1) = m-1+1 = m$

When m is even

$fog(m) = f(g(m)) = f(m+1) = m+1-1 = m$

$\therefore gof = I_W$ and $fog = I_W$

f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f.
inverse of f is f itself.

Question 3:

If $f: R \to R$ be defined as $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Solution:

$f: R \to R$ is defined as $f(x) = x^2 - 3x + 2$.

$$f(f(x)) = f(x^2 - 3x + 2)$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$
$$= (x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2) + (-3x^2 + 9x - 6) + 2$$
$$= x^4 - 6x^3 + 10x^2 - 3x$$

Question 4:

Show that function $f: R \to \{x \in R: -1 < x < 1\}$ be defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$ is one-one and onto function.

Solution:

$f: R \to \{x \in R: -1 < x < 1\}$ is defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$.

For one-one:

$f(x) = f(y)$ where $x, y \in R$

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

If x is positive and y is negative,

$$\frac{x}{1+|x|} = \frac{y}{1+|y|}$$
$$\Rightarrow 2xy = x - y$$

Since, x is positive and y is negative,

$x > y \Rightarrow x - y > 0$

$2xy$ is negative.

$2xy \neq x - y$

Case of x being positive and y being negative, can be ruled out.

$\therefore$ x and y have to be either positive or negative.

If x and y are positive,

$$f(x) = f(y)$$
$$\Rightarrow \frac{x}{1+x} = \frac{y}{1+y}$$
$$\Rightarrow x - xy = y - xy$$
$$\Rightarrow x = y$$

$\therefore f$ is one-one.

For onto:

Let $y \in R$ such that $-1 < y < 1$.

If x is negative, then there exists $x = \frac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If x is positive, then there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x)=f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|}=\frac{\frac{y}{1-y}}{1+\left(\frac{y}{1-y}\right)}=\frac{y}{1-y+y}=y$$

$\therefore f$ is onto.

Hence, f is one-one and onto.

Question 5:

Show that function $f: R \to R$ be defined by $f(x)=x^3$ is injective.

Solution:

$f: R \to R$ is defined by $f(x)=x^3$

For one-one:

$f(x)=f(y)$ where $x, y \in R$

$x^3 = y^3$........................(1)

We need to show that $x=y$

Suppose $x \neq y$, their cubes will also not be equal.

$\Rightarrow x^3 \neq y^3$

This will be a contradiction to (1).

$\therefore x=y$. Hence, f is injective.

Question 6:

Give examples of two functions $f: N \to Z$ and $g: Z \to Z$ such that gof is injective but g is not injective.

(Hint: Consider $f(x)=x$ and $g(x)=|x|$)

Solution:

Define $f: N \to Z$ as $f(x)=x$ and $g: Z \to Z$ as $g(x)=|x|$

Let us first show that g is not injective.

$(-1)=|-1|=1$

$(1)=|1|=1$

$\therefore (-1)=g(1), \quad \text{but } -1 \neq 1$

$\therefore g$ is not injective.

$gof : N \to Z$ is defined as $gof(x) = g(f(x)) = g(x) = |x|$

$x, y \in N$ such that $gof(x) = gof(y)$

$\Rightarrow |x| = |y|$

Since $x, y \in N$, both are positive.

$\therefore |x| = |y|$

$\Rightarrow x = y$

$\therefore gof$ is injective.

Question 7:

Given examples of two functions $f : N \to N$ and $g : N \to N$ such that gof is onto but f is not onto.

(Hint: Consider $f(x) = x+1$ and $g(x) = \begin{cases} x-1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$)

Solution:

Define $f : N \to Z$ as $f(x) = x+1$ and $g : Z \to Z$ as $g(x) = \begin{cases} x-1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$

Let us first show that g is not onto.

Consider element 1 in co-domain N. This element is not an image of any of the elements in domain N.

$\therefore f$ is not onto.

$g : N \to N$ is defined by

$gof(x) = g(f(x)) = g(x+1) = x+1-1 = x$ $[x \in N \Rightarrow x+1 > 1]$

For $y \in N$, there exists $x = y \in N$ such that $gof(x) = y$.

$\therefore gof$ is onto.

Question 8:

Given a non-empty set X, consider $P(X)$ which is the set of all subsets of X.

Define the relation R in $P(X)$ as follows:

For subsets A, B in $P(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $P(X)$? Justify you answer.

Solution:

Since every set is a subset of itself, ARA for all $A \in P(X)$.
$\therefore$ R is reflexive.

Let $ARB \Rightarrow A \subset B$
This cannot be implied to $B \subset A$.
If $A = \{1,2\}$ and $B = \{1, 2,3\}$, then it cannot be implied that B is related to A.
$\therefore$ R is not symmetric.

If ARB and BRC, then $A \subset B$ and $B \subset C$.
$\Rightarrow A \subset C$
$\Rightarrow ARC$
$\therefore$ R is transitive.

R is not an equivalence relation as it is not symmetric.

Question 9:

Given a non-empty set X, consider the binary operation $*$: $\mathrm{P}(X) \times \mathrm{P}(X) \rightarrow \mathrm{P}(X)$ given by $A * B = A \cap B \ \forall A, B$ in $P(X)$ is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in $P(X)$ with respect to the operation $*$.

Solution:

$\mathrm{P}(X) \times \mathrm{P}(X) \rightarrow \mathrm{P}(X)$ given by $A * B = A \cap B \ \forall A, B$ in $P(X)$

$A \cap X = A = X \cap A$ for all $A \in P(X)$

$\Rightarrow A * X = A = X * A$ for all $A \in P(X)$
X is the identity element for the given binary operation $*$.

An element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that
$A * B = X = B * A$ [As X is the identity element]

Or

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

X is the only invertible element in $P(X)$ with respect to the given operation $*$.

Question 10:

Find the number of all onto functions from the set $\{1,2,3,\ldots,n\}$ to itself.

Solution:

Onto functions from the set $\{1,2,3,\ldots,n\}$ to itself is simply a permutation on n symbols $1,2,3,\ldots,n$.

Thus, the total number of onto maps from $\{1,2,3,\ldots,n\}$ to itself is the same as the total number of permutations on n symbols $1,2,3,\ldots,n$, which is $n!$.

Question 11:

Let $S=\{a,b,c\}$ and $T=\{1,\ 2,3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

i. $F=\{(a,3),(b,2),(c,1)\}$
ii. $F=\{(a,2),(b,1),(c,1)\}$

Solution: $S=\{a,b,c\},T=\{1,\ 2,3\}$

i. $F:S\rightarrow T$ is defined by $F=\{(a,3),(b,2),(c,1)\}$
$\Rightarrow F(a)=3,\ F(b)=2,\ F(c)=1$
Therefore, $F^{-1}:T\rightarrow S$ is given by $F^{-1}=\{(3,a),(2,b),(1,c)\}$

ii. $F:S\rightarrow T$ is defined by $F=\{(a,2),(b,1),(c,1)\}$
Since, $F(b)=F(c)=1$, F is not one-one.
Hence, F is not invertible i.e., F^{-1} does not exists.

Question 12:

Consider the binary operations $*$: $R\times R\rightarrow R$ and $o:R\times R\rightarrow R$ defined as $a*b=|a-b|$ and $aob=a,\ \forall a,b\in R$. Show that $*$ is commutative but not associative o is associative but not commutative. Further, show that $\forall a,b,c\in R$, $a*(boc)=(a*b)o(a*c)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

Solution:

It is given that $*$: $R\times R\rightarrow R$ and $o:R\times R\rightarrow R$ defined as $a*b=|a-b|$ and $aob=a,\ \forall a,b\in R$.

For $a,b\in R$, we have $a*b=|a-b|$ and $b*a=|b-a|=|-(a-b)|=|a-b|$

$\therefore a*b=b*a$

$\therefore$ The operation $*$ is commutative.

$$(1*2)*3=(|1-2|)*3=1*3=|1-3|=2$$
$$1*(2*3)=1*(|2-3|)=1*1=|1-1|=0$$

$\therefore (1*2)*3 \neq 1*(2*3)$ where $1,2,3 \in R$

$\therefore$ The operation * is not associative.

Now, consider the operation o:

It can be observed that $1o2=1$ and $2o1=2$.

$\therefore 1o2 \neq 2o1$ (where $1,2 \in R$)

$\therefore$ The operation o is not commutative.

Let $a,b,c \in R$. Then, we have:

$$(aob)oc = aoc = a$$
$$ao(boc) = aob = a$$
$$\Rightarrow (aob)oc = ao(boc)$$

$\therefore$ The operation o is associative.

Now, let $a,b,c \in R$, then we have:

$$a*(boc) = a*b = |a-b|$$
$$(a*b)o(a*c) = (|a-b|)o(|a-c|) = |a-b|$$

Hence, $a*(boc) = (a*b)o(a*c)$

Now,

$$1o(2*3) = 1o(|2-3|) = 1o1 = 1$$
$$(1o2)*(1o3) = 1*1 = |1-1| = 0$$

$\therefore 1o(2*3) \neq (1o2)*(1o3)$ where $1,2,3 \in R$

$\therefore$ The operation o does not distribute over $*$.

Question 13:

Given a non - empty set X, let $*$: $P(X)\times P(X)\rightarrow P(X)$ be defined as $A*B=(A-B)\cup(B-A)$, $\forall A,B\in P(X)$. Show that the empty set Φ is the identity for the operation $*$ and all the elements A of $P(X)$ are invertible with $A^{-1}=A$.

(Hint: $(A-\Phi)\cup(\Phi-A)=A$ and $(A-A)\cup(A-A)=A*A=\Phi$).

Solution:

It is given that $*$: $P(X)\times P(X)\rightarrow P(X)$ is defined as $A*B=(A-B)\cup(B-A)$, $\forall A,B\in P(X)$

$A\in P(X)$ then,

$A*\Phi=(A-\Phi)\cup(\Phi-A)=A\cup\Phi=A$

$\Phi*A=(\Phi-A)\cup(A-\Phi)=\Phi\cup A=A$

$\therefore A*\Phi=A=\Phi*A$ for all $A\in P(X)$

Φ is the identity for the operation $*$.

Element $A\in P(X)$ will be invertible if there exists $B\in P(X)$ such that

$A*B=\Phi=B*A$ [As Φ is the identity element]

$A*A=(A-A)\cup(A-A)=\Phi\cup\Phi=\Phi$ for all $A\in P(X)$.

All the elements A of $P(X)$ are invertible with $A^{-1}=A$.

Question 14:

Define a binary operation $*$ on the set $\{0,1,2,3,4,5\}$ as

$$a+b=\begin{cases}a+b, & \text{if } a+b<6\\ a+b-6 & \text{if } a+b\geq 6\end{cases}$$

Show that zero is the identity for this operation and each element $a\neq 0$ of the set is invertible with $6-a$ being the inverse of a.

Solution:

Let $X=\{0,1,2,3,4,5\}$

The operation $*$ is defined as $$a+b=\begin{cases}a+b, & \text{if } a+b<6\\ a+b-6, & \text{if } a+b\geq 6\end{cases}$$

An element $e\in X$ is the identity element for the operation $*$, if $a*e=a=e*a \quad \forall a\in X$

For $a\in X$,

$a * 0 = a + 0 = a \qquad [a \in X \Rightarrow a + 0 < 6]$

$0 * a = 0 + a = a \qquad [a \in X \Rightarrow 0 + a < 6]$

$\therefore a * 0 = a = 0 * a \quad \forall a \in X$

Thus, 0 is the identity element for the given operation $*$.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

i.e., $\begin{Bmatrix} a + b = 0 = b + a, & \text{if } a + b < 6 \\ a + b - 6 = 0 = b + a - 6 & \text{if } a + b \geq 6 \end{Bmatrix}$

$\Rightarrow a = -b$ or $b = 6 - a$

$X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$. Then $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a for all $a \in X$.
Inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a - 1 = 6 - a$.

Question 15:

Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g : A \to B$ be functions defined by $x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in A$. Are f and g equal?

Solution:

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$

Also, $f, g : A \to B$ is defined by $x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in A$.

$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$

$g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$

$\Rightarrow f(-1) = g(-1)$

$f(0) = (0)^2 - 0 = 0$

$g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\Rightarrow f(0) = g(0)$

$f(1)=(1)^2-1=0$

$g(1)=2\left|1-\frac{1}{2}\right|-1=2\left(\frac{1}{2}\right)-1=1-1=0$

$\Rightarrow f(1)=g(1)$

$f(2)=(2)^2-2=2$

$g(2)=2\left|2-\frac{1}{2}\right|-1=2\left(\frac{3}{2}\right)-1=3-1=2$

$\Rightarrow f(2)=g(2)$

$\therefore f(a)=g(a) \quad \forall a \in A$

Hence, the functions f and g are equal.

Question 16:

Let $A=\{1,2,3\}$. Then number of relations containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is,

A. 1
B. 2
C. 3
D. 4

Solution:

The given set is $A=\{1,2,3\}$.

The smallest relation containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is given by,

$R=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(3,1)\}$

This is because relation R is reflexive as $\{(1,1),(2,2),(3,3)\}\in R$.

Relation R is symmetric as $\{(1,2),(2,1)\}\in R$ and $\{(1,3)(3,1)\}\in R$.

Relation R is transitive as $\{(3,1),(1,2)\}\in R$ but $(3,2)\notin R$.

Now, if we add any two pairs $(3,2)$ and $(2,3)$ (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 17:

Let $A=\{1,2,3\}$. Then number of equivalence relations containing $(1,2)$ is,

A. 1
B. 2
C. 3
D. 4

Solution:

The given set is $A=\{1,2,3\}$.

The smallest equivalence relation containing $(1,2)$ is given by;

$R_1=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

Now, we are left with only four pairs i.e., $(2,3),(3,2),(1,3)$ and $(3,1)$.

If we odd any one pair [say $(2,3)$] to R_1, then for symmetry we must add $(3,2)$. Also, for transitivity we are required to add $(1,3)$ and $(3,1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing $(1,2)$ is two.
The correct answer is B.

Question 18:

Let $f:R\to R$ be the Signum Function defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$ and $g:R\to R$ be the greatest integer function given by $g(x)=[x]$, where $[x]$ is greatest integer less than or equal to x. Then does fog and gof coincide in $(0,1]$?

Solution:

It is given that $f:R\to R$ be the Signum Function defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$

Also $g:R\to R$ is defined as $g(x)=[x]$, where $[x]$ is greatest integer less than or equal to x.
Now let $x\in(0,1]$,

$[x]=1$ if $x=1$ and $[x]=0$ if $0<x<1$.

$$\therefore fog(x) = f(g(x)) = f([x]) = \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0,1) \end{cases} = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0,1) \end{cases}$$

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(1) \qquad [x > 0] \\ &= [1] = 1 \end{aligned}$$

Thus, when $x \in (0,1)$, we have $fog(x) = 0$ and $gof(x) = 1$.

Hence, fog and gof does not coincide in $(0,1]$.

Question 19:

Number of binary operations on the set $\{a,b\}$ are

A. 10
B. 16
C. 20
D. 8

Solution:

A binary operation * on $\{a,b\}$ is a function from $\{a,b\} \times \{a,b\} \to \{a,b\}$

i.e., * is a function from $\{(a,a),(a,b),(b,a),(b,b)\} \to \{a,b\}$

Hence, the total number of binary operations on the set $\{a,b\}$ is $2^4 = 16$.

The correct answer is B.